

Theorem 1.2 (Chen). *Every sufficiently large even number N can be expressed as*

$$N = p + \eta, \quad (1.2)$$

where p is prime and $\eta > 0$ has at most 2 prime factors.

Theorem 1.3 (Li). *Every sufficiently large odd number N can be expressed as*

$$N = p + 2\eta, \quad (1.3)$$

where p is prime and $\eta > 0$ has at most 2 prime factors.

Here, the factor of 2 in (1.3) is to force $p + 2\eta$ to be odd; provided of course $p \neq 2$.

Combining Theorems 1.2 and 1.3, we have that every large $N \geq 1$ can be represented as the sum of a prime and a number with at most 3 prime factors. Any further lowering of the number of prime factors in Chen or Li's results appears out of reach, so we do not attempt this here.

Now, representing N of the form (1.1) when $k \geq 2$ is naturally more difficult, owing to sparseness of the set of k th prime powers, and additional “forced” prime factors of $N - p^k$. Thus in this setting, much less is known. However, the following classical result, due to Erdős [7] and Rao [24], can be viewed as partial progress towards representing large N in the form (1.1).

Theorem 1.4 (Erdős–Rao). *Let $k \geq 2$. Then every sufficiently large integer N such that*

$$N \not\equiv 1 \begin{cases} \pmod{16}, & \text{when } k = 4, \\ \pmod{4}, & \text{when } k = 2, \end{cases} \quad (1.4)$$

can be expressed as

$$N = p^k + \eta$$

where p is prime and $\eta > 0$ is k th power-free.

Here, the condition that η is k th power-free is much simpler than η having a bounded number of prime factors. In particular, most integers ($> 60\%$) are square-free (and thus k th power-free), whereas the set of integers with a bounded number of prime factors has a natural density of 0.

1.2 Statement of results

Our main result is as follows.

Theorem 1.5. *Let $k \geq 2$. There exists an integer $M(k)$ such that every sufficiently large integer N can be expressed as*

$$N = p^k + \eta \quad (1.5)$$

where p is prime and $\eta > 0$ has at most $M(k)$ prime factors. Here, $M(k) = 6k$ is admissible for all even $k \geq 2$, and $M(k) = 4k$ is admissible for all odd $k \geq 3$. In addition, for sufficiently large $k \geq k_\varepsilon$ one can set

$$M(k) = (2 + \varepsilon)k, \quad (1.6)$$

for any $\varepsilon > 0$. Or, under assumption of the Elliott–Halberstam conjecture¹,

$$M(k) = (1 + \varepsilon)k. \quad (1.7)$$

¹More precisely, a standard variant of the Elliott–Halberstam conjecture (see Conjecture 2.7).